10/22 Recitation Solutions

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of nonzero orthogonal vectors. Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

   Suppose $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ for some weights $x_1, \ldots, x_n$. We need to show that $x_i = 0$ for all i. Taking the dot product of this equation with $\mathbf{v}_i$ yields

   $$\mathbf{v}_i \cdot (x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) = \mathbf{v}_i \cdot \mathbf{0}$$
   $$x_1(\mathbf{v}_i \cdot \mathbf{v}_1) + \cdots + x_n(\mathbf{v}_i \cdot \mathbf{v}_n) = 0$$

   Since $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, the equation reduces to $x_i\mathbf{v}_i \cdot \mathbf{v}_i = 0$. Now since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\mathbf{v}_i \cdot \mathbf{v}_i \neq 0$ and so $x_i = 0$. Hence $x_i = 0$ for all i and so the vectors are linearly independent.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$.

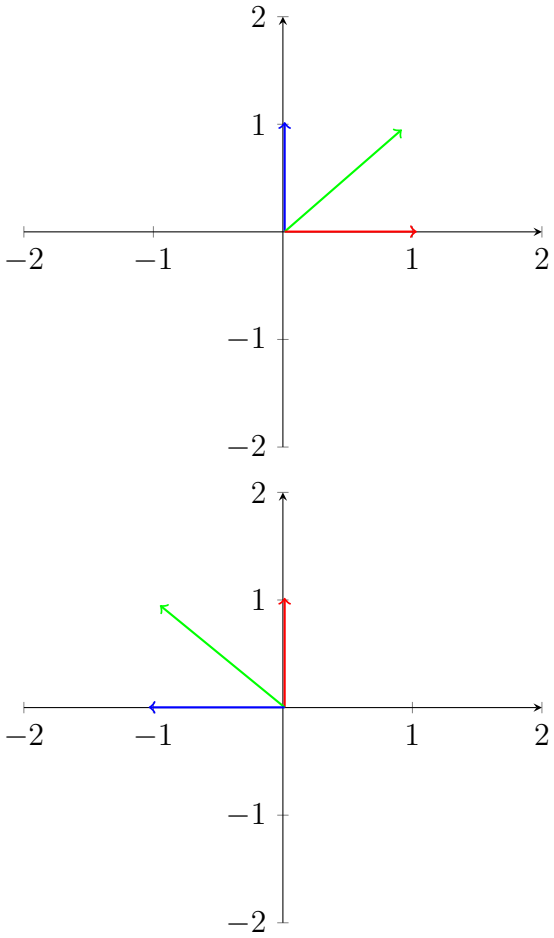
   (a) Find the matrix associated to T and show that it is orthogonal.

   The associated matrix is $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ because $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$. Since Q is square, the two columns of Q are unit vectors and their dot product is 0, Q is othogonal.

   (b) Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and sketch the original vectors and images in $\mathbb{R}^2$.

   $$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
   $$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
   $$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   The input and corresponding output vectors are drawn below in red, green, and blue, respectively.



(c) What does T do to these vectors, geometrically? Use this intuition to explain why T preserves lengths and angles.

Notice that they vectors are obtained from their inputs by rotating them by 90°. In general, T rotates in $\mathbb{R}^2$ vectors by 90°. Since rotation doesn't change angles between vectors or lengths of vectors, T preserves lengths and angles.

3. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

(a) Use Gram-Schmidt to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ of $C(A)$.

We first find an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2\}$.

$\mathbf{b}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\mathbf{b}_2 = \mathbf{a}_2 - \text{proj}_{\mathbf{b}_1}\mathbf{a}_2 = \mathbf{b}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{-9}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Next, we normalize these vectors to obtain an orthonormal basis:

$\mathbf{q}_1 = \frac{1}{||\mathbf{b}_1||}\mathbf{b}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\mathbf{q}_2 = \frac{1}{||\mathbf{b}_2||}\mathbf{b}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

(b) Use part (a) to find a QR factorization of A.

$Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}$

$R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$

Hence $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$

(c) Use the QR factorization to find a least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We can find a least squares solution by solving $R\mathbf{x} = Q^T\mathbf{b}$;
that is solve $\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. Since R is upper triangular, we can use back substitution, which gives the solution $\hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 2/9 \end{bmatrix}$.